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Transport and Wigner delay time distribution across a random active medium

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Abstract : We study the wave propagation through a single-channel (single-mode) coherently amplifying disordered medium. A new crossover length scale is introduced in the regime of strong disorder and weak amplification. We show that in an active medium reflectance arises due to synergetic effect of localization and coherent amplification. Our study reveals that the tail of the Wigner delay time distribution from a disordered passive medium exhibits a universality in the sense that it is independent of the nature of disorder.

Keywords : Disorder, amplification, localization, delay time

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Wave propagation in passive disordered media continues to be a subject of great interest [1]. Simple models of wave/particle moving in a random potential can be used to describe such variety of phenomena as Anderson localization, photon or light localization in a random dielectric medium [2], sound propagation in inhomogenous media, *etc*. These waves, though qualitatively different, obey the Helmholtz equation in appropriate limit. The common operative feature is the interference and diffraction of waves.

In recent years the subject of wave propagation in an active random medium, *i.e.*, in the presence of amplification/absorption, has attracted considerable attention [3–5]. Light wave propagation through a spatially random but laser-active (amplifying) dielectric medium is an excellent laboratory for studying the interplay between disorder-induced localization (Anderson localization) and coherent amplification. To describe the amplification/absorption complex potentials are used leading to non-Hermitian Hamiltonians and hence non-conservation of particle number. It is worthwhile to note that the temporal coherence of wave is preserved in spite of the amplification/absorption. In present work we will be concerned with two aspects of the transport through one-dimensional disordered systems, namely, the statistics of transmission and reflection in presence of coherent amplification and the universality of the tail of the Wigner delay time distribution.

The dual role played by an imaginary potential as an amplifier/absorber and as a reflector has been emphasized in Ref. [6]. Using duality relations it has been shown that the amplification suppresses the transmittance in the large length limit just as much as absorption does irrespective of the strength of the disorder [7]. Even though the transmittance decreases exponentially in the asymptotic limit, the transmission coefficient (t) is a non-self-averaging quantity but with a finite well-defined average value [5]. This is in contradiction with the naive expectation of (t) being infinite owing to the contribution from the resonant states. However, the fact that even for the case of no disorder (all states resonant) asymptotically $t \to 0$ clarifies this ambiguity [5]. There exists a crossover length L_c, below which the amplification enhances transmission and above which the amplification reduces the transmission which vanishes exponentially in the $L \to \infty$ limit. The length L_c was shown [8] to behave like $1/W\sqrt{\eta}$, where W is the strength of disorder and η is the strength of amplification. This suggests that as $W \to 0$, L_c would tend to infinity. This is in contradiction with the analytical result which clearly shows that L_c is finite and non-zero even for $W = 0$ case. Evidently the result $L_c \sim 1/W\sqrt{\eta}$ is valid only in certain region of the parameter space. To investigate this, we consider the following single-band tight-binding Hamiltonian to model the motion of a qausi-particle moving on a lattice [4,5] :

$$H = \sum \varepsilon_n' |n\rangle\langle n| + V\big(|n\rangle\langle n+1| + |n\rangle\langle n-1|\big). \qquad (1)$$

V is the off-diagonal matrix element connecting nearest neighbors separated by a lattice spacing a (taken to be unity throughout) and $|n\rangle$ is the non-degenerate Wannier orbital associated with site n, where $\varepsilon_n' = \varepsilon_n - i\eta$ is the site energy. The real part of the site energy ε_n being random represents static disorder and ε_n at different sites are assumed to be uncorrelated random variables distributed uniformly ($P(\varepsilon_n) = 1/W$) over the range $-W/2$ to $W/2$. We have taken imaginary part of the site energy η to be spatially uniform positive variable for amplification. Since all the relevant energies can be scaled by V, we can set V to unity. The lasing medium consisting of N sites (n = 1 to N) is embedded in a perfect infinite lattice with all site energies taken to be zero. To calculate the transmission and reflection coefficients we use the well known transfer-matrix method, and the details are described in Ref. [4,5].

In our studies we have set the energy of the incident particle at E = 0, *i.e.*, at a midband energy. Any other value for the incident energy does not affect the physics of the problem. In calculating average values in all cases we have taken 10,000 realizations of random site energies (ε_n). The strength of the disorder and the amplification are scaled with respect to V, *i.e.*, W ($\equiv W/V$) and η ($\equiv \eta/V$) The length $L = L/a$.

In Figure 1 we have plotted <*lnt*> against L for ordered lasing medium ($W = 0$, $\eta = 0.01$), disordered passive medium ($W = 1.0$, $\eta = 0$) and disordered active medium ($W = 1.0$, $\eta = 0.01$). The present study is restricted to the parameter space of η and W such that $\eta \ll 1.0$ and $W \geq 1.0$. We notice that for an ordered lasing medium, the transmittance is larger

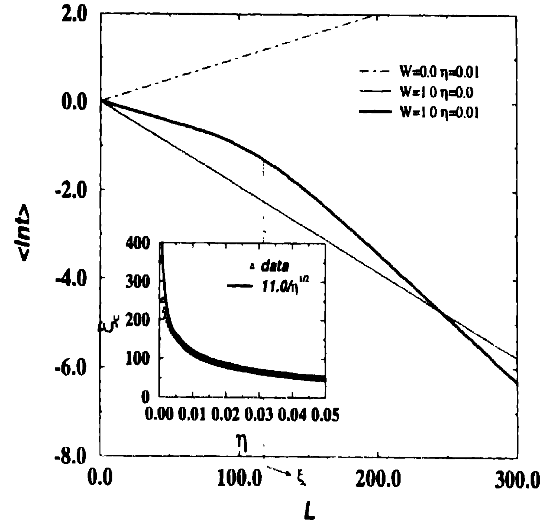

Figure 1. Variation of <*lnt*> with L. The new length scale ξ_c which arises for $\eta \ll 1.0$ is shown by a vertical dotted line. The inset shows the variation of ξ_c with η for $W = 1.0$. The numerical fit shown by the thick line indicates that ξ_c scales as $\eta^{-1/2}$ in this regime.

than one. We have taken our range of L upto 300. For a disordered active medium ($W = 1.0$, $\eta = 0.01$), we notice that the transmittance is always less than one and monotonically decreasing. Initially, upto certain length, the average transmittance is, however, larger than that in the disordered passive medium ($W = 1.0$, $\eta = 0$). This arises due to the combination of lasing with disorder. In the asymptotic regime transmittance of a lasing random medium falls below that in the passive medium with same disorder strength. This follows from the enhanced localization effect due to the presence of both disorder and amplification together, *i.e.*, $\xi < l$ where ξ is the localization length in the presence of both disorder and amplification and l is the localization length due to disorder alone. It is clear from the figure that <*lnt*> does not exhibit any maxima and hence the question of L_c does not arise. We notice, however, from the figure that for random active medium initially <*lnt*> decreases with a well defined slope and in the large length limit <*lnt*> decreases with a different slope (corresponding to localization length ξ). Thus we can define a length scale ξ_c (as indicated in the figure) at which there is a cross-over from the initial slope to the asymptotic slope. In the inset of Figure 1 we have shown the dependence of ξ_c on η. Numerical fit shows that ξ_c scales as $1/\sqrt{\eta}$, as we expect $\xi_c \to \infty$ with $\eta \to 0$. As one decreases η, the absolute value of initial slope increases and that of the asymptotic one decreases. Simultaneously, the cross-over length ξ_c increases. In the $\eta \to 0$ limit both initial as well as asymptotic slopes become identical.

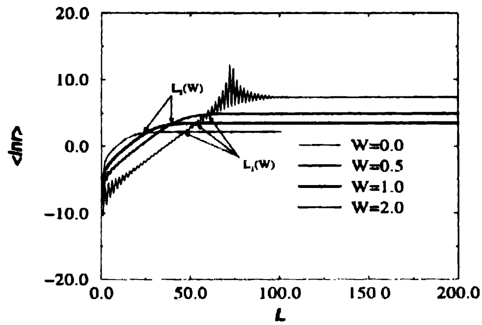

Figure 2. Variation of <lnr> with L for values of W indicated in the figure The two length scales $L_1(W)$ and $L_2(W)$ associated with the reflectance are shown with arrows.

We would now like to understand the role of interplay between Anderson localization and coherent amplification in enhancement of the reflection. In Figure 2 we plot <lnr> as a function of the length L for a fixed value of amplification strength $\eta = 0.1$ and for various values of the disorder strength W as indicated in the figure. In the absence of disorder ($W = 0$) as one varies length, initially the reflectance increases to a very large value through large oscillations and after exhibiting a maximum again through oscillations, it eventually saturates to a finite (large) value. In the presence of disorder one can readily notice that initially <lnr> increases and has a magnitude larger than that for $W = 0$ case and asymptotically beyond a disorder dependent length scale $L_1(W)$, it saturates to a value which is smaller than that for a $W = 0$ case. The saturation value of <lnr> decreases as one increases the disorder as a result of localization induced by combined effect of disorder and amplification. Below the length scale $L_1(W)$ we identify another disorder dependent length scale $L_2(W)$. Above L_2 (but smaller than L_1) further increase in disorder suppresses the reflectance whereas below L_2 it enhances the reflectance. The length scale L_2 being much smaller than the localization length l for the passive medium, increase in disorder causes multiple reflections in a sample of size smaller than L_2 and due to the increase in delay time we get enhanced back reflection. Beyond L_2 due to disorder induced localization delay time decreases and as a consequence we obtain reduced reflectance.

We now dissertate on the issue of the universality of the tail of the distribution of Wigner delay time of a passive one-dimensional random medium. The delay time in the scattering process is generally taken to be related to the duration of a collision event or time spent by the particle in the region of interaction. The delay time statistics is intimately connected with the dynamic admittance of microstructures. For a single channel the distribution of the delay time for a disordered semi-infinite sample has been obtained earlier by using the invariant imbedding approach [9]. The stationary distribution $P_s(\tau)$ for the dimensionless delay time τ is given by

$$P_s(\tau) = \frac{\lambda e^{\lambda \tan^{-1}\tau}}{(e^{\lambda\pi/2} - 1)(1+\tau^2)}, \tag{2}$$

where λ is proportional to the disorder induced localization length and the most probable value of τ occurs at $\tau_{max} = \lambda/2$. The long time tail of the above distribution scales as $1/\tau^2$. The average value of τ is logarithmically divergent indicating the possibility of the particle traversing the infinite sample before being totally reflected, due to the resonances. If the disordered region is semi-infinite, the reflection coefficient will be unity, and the complex reflection amplitude will have the form $R = e^{i\theta(E)}$. If the wave packet is incident on the disordered sample it will not be immediately reflected back into the lead region, but will be delayed by time proportional to $\tau = \hbar d\theta/dE$. This energy dependent random time delay leads to a non-cancellation of the instantaneous currents at the surface involving the incident and reflected particles. This is expected to lead to a low temperature $1/f$ type noise that should be universal [9]. A very recent study based on analytical work found the delay time distribution in the one-channel case to be universal (especially the long time tail is independent of the nature of disorder) [10]. We would like to examine this through our study. We would like to emphasize that, in order to obtain $P_s(\tau)$ (eqn. 2) earlier studies invoke several approximations such as the random phase approximation (RPA), which is only valid in the small disorder regime and moreover, the correlation between the phase and the delay time is neglected.

In order to calculate the reflection amplitude we use the same model as described above except for the disorder distribution. We consider three kinds of disorder where the site energies ε_n are assumed uncorrelated random variables having distributions which are uniform ($P(\varepsilon_n) = 1/W$), Gaussian ($P(\varepsilon_n) \propto e^{-\varepsilon_n^2/2W^2}$) and exponential ($P(\varepsilon_n) \propto e^{-\varepsilon_n/W}$). The transfer-matrix method [4,5] is used to calculate the reflection amplitude $r(E) = |r|e^{-i\theta(E)}$ and its phase $\theta(E)$ at two values of incident energy $E = E_0 \pm \delta E$. The delay time is then calculated using the definition $\tau = \hbar d\theta/dE$. Throughout our following discussion we consider the delay time τ in a dimensionless form by multiplying it with V and we set $\hbar = m = 1$. In view of the fact that the value of the incident energy E_0 will not change the physics of the problem, in the following we choose $E_0 = 0$ and $dE = 2\delta E = 0.002$. In calculating the stationary distribution of delay time we take at least 10^6 realizations of a disordered sample of length (L) equal to 8 times the localization length (ξ), where the localization length is calculated by a standard prescription [4,5].

In Figure 3(a) and (b) we show the numerical data (thin line) for the stationary distribution $P_s(\tau)$ of the delay time τ for weak disorder ($W = 0.5$) and strong disorder ($W = 2.0$) respectively. The thick line in the figure is the numerical fit obtained by using the expression for $P_s(\tau)$ given in eqn. 2. We see that the fit is fairly good even for strong disorder ($W = 2.0$) for which the stationary distribution of the phase of the reflected wave,

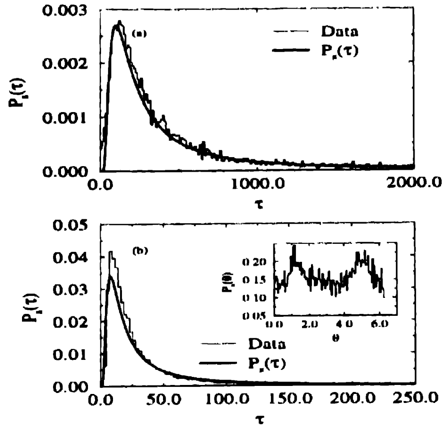

Figure 3. The stationary distribution of delay time $P_s(\tau)$ for (a) weak disorder ($W = 0.5$) and (b) strong disorder ($W = 2.0$).

$P_s(\theta)$, shows (inset of Figure 3(b)) two distinct peaks indicating the failure of the RPA in this regime.

We now look at the tail of the delay distribution and its universality for the three kinds of disorder beyond RPA. Since the origin of the tail is due to the appearance of

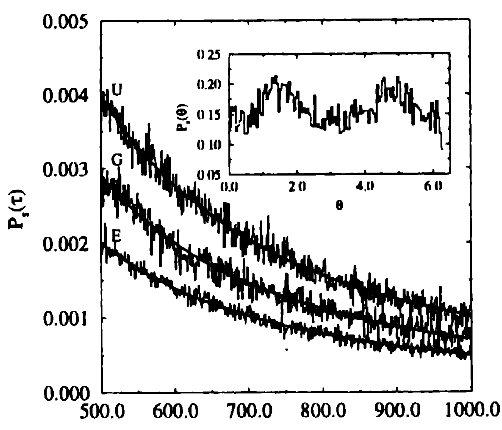

Figure 4. The plot of tail of $P_s(\tau)$ for the case of uniform (U), Gaussian (G) and exponential (E) disorder The disorder strength in all the three cases is $W = 1.0$ The plots have been shifted on the Y-axis to avoid overlap which would obscure the details.

resonant realizations which are independent of strength and the type of disorder, we expect that the tail distribution would be universal beyond RPA. In Figure 4 we plot the tail distribution of $P_s(\tau)$ for uniform, Gaussian and exponential disorder characterized by the strength $W = 1.0$. The numerical, least-square fit for the expression α / τ^β to the long-time tail data gives $\beta \approx 2$ for all the cases. The values of exponent β for the different kinds of disorder and different strengths of disorder are summarized in Table 1. For the value

Table 1. The values of exponent β obtained by least-square fit for the expression α/τ^β to the data for different kinds of disorder and different strengths of disorder.

Kind of disorder	β for $W = 1.0$	β for $W = 1.5$
Uniform (U)	1.979	2.006
Gaussian (G)	2.047	1 987
Exponential (E)	2.024	1.961

$W = 1.0$, we are in a regime beyond RPA as can be seen from the non-uniformity of the stationary distribution $P_s(\theta)$ of the phase of the reflected wave shown in the inset of the Figure 4. For the stronger disorder case of $W = 1.5$ also we obtain the value of exponent β to be 2. Therefore, our numerical simulation results suggest the existence of universality in the long time tail distribution.

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